

A CLASS OF GENERALIZED POSITIVE LINEAR MAPS ON MATRIX ALGEBRAS

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ABSTRACT. We construct a class of positive linear maps on matrix algebras. We find conditions when these maps are atomic, decomposable and completely positive. We obtain a large class of atomic positive linear maps. As applications in quantum information theory, we discuss the structural physical approximation and optimality of entanglement witness associated with these maps.

1. INTRODUCTION

Positive linear maps on C^* -algebras, particularly those of finite dimensions, have been becoming more important by their connection with quantum information theory. A linear map on a C^* -algebra is called *positive* if it sends the cone of positive elements into itself. Little is known about the global structure of positive linear maps, even in the low dimensional matrix algebras. Let M_n be the C^* -algebra of all $n \times n$ matrices over the complex field, and let $\mathcal{P}_k(M_n)$ (respectively, $\mathcal{P}^k(M_n)$) be the convex cone of all k -positive (respectively, k -copositive) linear maps on M_n . One of the basic problems about the structures of the positive cone $\mathcal{P}_1(M_n)$ is whether the set $\mathcal{P}_1(M_n)$ can be decomposed as the algebraic sum of some simpler classes in $\mathcal{P}_1(M_n)$ [23]. When $n = 2$, it is well known [24] that every positive linear map can be written as a sum of a completely positive linear map and a completely copositive linear map, that is, the maps in $\mathcal{P}_1(M_2)$ are decomposable. But this is not the case for higher dimensional matrix algebras. On M_3 , Choi gave an extremal positive linear map which is indecomposable [2]. Tanahashi and Tomiyama in [23] introduced the concept of atomic positive linear map which has a stronger indecomposability, and they showed that Choi's map is atomic. There are only a few examples of indecomposable positive linear maps in the literature, much less the atomic ones. Most known examples of indecomposable positive linear maps and atomic positive linear maps can be found in [3, 4, 25, 7, 13, 14] and references therein. In quantum information theory, indecomposable positive linear maps can be used to detect entangled states whose partial transposes are positive and atomic positive linear maps can be used to detect states with the 'weakest' entanglement [4]. Positive linear maps also play an important role in the study of operator system theory [19, 16], etc.

In this paper, we give a generalization of linear maps defined in [7]. Let S_n be the symmetric group consisting of all bijections (permutations) from the set $\{1, 2, \dots, n\}$ onto itself. For positive real numbers a, c_1, c_2, \dots, c_n and each $\sigma \in S_n$, we define a linear map

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$\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ from M_n to M_n by

$$\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n](X) = \Delta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n](X) - X,$$

where

$$\Delta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n](X) = \begin{pmatrix} ax_{11} + c_1 x_{\sigma(1),\sigma(1)} & 0 & \cdots & 0 \\ 0 & ax_{22} + c_2 x_{\sigma(2),\sigma(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ax_{nn} + c_n x_{\sigma(n),\sigma(n)} \end{pmatrix},$$

for each $X = (x_{ij}) \in M_n$. Let $\phi : M_n \mapsto M_n$ be a linear map. If it has the form

$$\phi : (x_{ij}) \mapsto \text{diag}(f_1, \dots, f_n) - (x_{ij}) \quad \text{with} \quad (f_1, \dots, f_n) = (x_{11}, \dots, x_{nn})D \quad (1.1)$$

where $D = (d_{ij})$ is an $n \times n$ nonnegative matrix, that is, all $d_{ij} \geq 0$, then ϕ is called a *D-type linear map* [12]. In (1.1), if we let

$$D = aI_n + \sum_{i=1}^n c_i E_{\sigma(i)i}$$

where I_n and $\{E_{ij}\}_{i,j=1}^n$ are the identity matrix and the canonical matrix units of M_n , respectively, we can see that $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ has the form in (1.1) and so it is a *D-type linear map*. Throughout this paper, if there is no confusion, $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ and $\Delta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ will often be abbreviated to $\Theta^{(n,\sigma)}$ and $\Delta^{(n,\sigma)}$, respectively.

For each $k \in \{1, 2, \dots, n\}$, we define $\tau_k^n \in S_n$ by

$$\tau_k^n(i) \equiv i + k \pmod{n}, \quad (1.2)$$

for $i = 1, 2, \dots, n$. The linear map $\Theta^{(3,\tau_2^3)}[a; c_1, c_2, c_3]$ was studied in [13]. In [7], Ha defined the map $\Theta^{(n,\tau_{n-1}^n)}[a; c_1, c_2, \dots, c_n]$ which is a generalization of $\Theta^{(3,\tau_2^3)}[a; c_1, c_2, c_3]$ and gave a sufficient condition for the map $\Theta^{(n,\tau_{n-1}^n)}[a; c_1, c_2, \dots, c_n]$ being atomic. In [20], Qi and Hou defined the map $\Theta^{(n,\tau_k^n)}[n-1; 1, 1, \dots, 1]$ and discussed when $\Theta^{(n,\tau_k^n)}$ is positive and indecomposable. In [21], Qi and Hou studied the optimality, decomposability and structural physical approximation of $\Theta^{(n,\tau_k^n)}[n-1; 1, 1, \dots, 1]$ for $k \neq n$. For each $\sigma \in S_n$ and $c \geq 0$, the positivity of $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ was discussed in [12]. For $\sigma^2 = id_n$ where id_n is the identity of S_n , the decomposability of $\Theta^{(n,\sigma)}[n-1; 1, 1, \dots, 1]$ was also discussed in [12]. In [8], Ha discussed the optimality of the entanglement witness associated with $\mathbf{T} \circ \Theta^{(n,\tau_k^n)}[n-1; 1, 1, \dots, 1]$ for $k \neq n$ and $\frac{n}{2}$ (when n ($n \geq 3$) is even), where \mathbf{T} denotes the transpose map.

The paper is organized as follows. In Section 2 we give conditions when the map $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is positive and discuss the equivalence between 2-positivity and completely positivity. In Section 3 we give conditions when $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is atomic and decomposable. We give conditions in Section 4 when the structural physical approximation of $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is separable and the entanglement witness associated with $\mathbf{T} \circ \Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is optimal.

Throughout this paper, a matrix A is positive means that A is positive semi-definite and is denoted by $A \geq 0$. For every vector in \mathbb{C}^n , we consider it as an $n \times 1$ matrix, that is, a column vector. If x is a vector or a matrix, then x' and x^* denote the transpose and conjugate transpose of x , respectively. Let $\{e_i\}_{i=1}^n$ and $\{E_{ij}\}_{i,j=1}^n$ denote the canonical orthonormal basis of \mathbb{C}^n and the matrix units of M_n , respectively. Let $\langle \cdot, \cdot \rangle$ be the usual inner product on \mathbb{C}^n and (n, k) denote the greatest common divisor of n and k . For $m, n \in \mathbb{N}$, if m divides n we

write $m|n$, and if m does not divide n we write $m \nmid n$. Let \mathbf{T} denote the transpose map on M_n and id_n denote the identity of S_n .

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2. POSITIVITY AND 2-POSITIVITY

In this section, we give conditions when $\Theta^{(n,\sigma)}$ is positive and then discuss the equivalence between 2-positivity and completely positivity.

Lemma 2.1. ([7]) *Let $a > 0$. For symmetric function*

$$F(x_1, \dots, x_n) = \sum_{m=0}^n \left[a^{m-1}(a-m) \sum_{1 \leq i_1 < \dots < i_{n-m} \leq n} x_{i_1} \cdots x_{i_{n-m}} \right],$$

where x_1, \dots, x_n are positive real numbers, we have that $F(x_1, \dots, x_n) \geq 0$ if and only if $\sum_{i=1}^n (a + x_i)^{-1} \leq 1$.

Lemma 2.2. ([7]) *For $x_i \geq 0$, $i = 1, 2, \dots, n$, $x \geq 0$ and real number a , we have the following:*

$$\sum_{1 \leq i_1 < \dots < i_{n-m} \leq n} x_{i_1} \cdots x_{i_{n-m}} \geq \frac{n!}{(n-m)!m!} (x_1 \cdots x_n)^{\frac{n-m}{n}}, \quad 0 \leq m < n; \quad (2.1)$$

$$(x^{\frac{1}{n}} + a)^{n-1} (x^{\frac{1}{n}} + a - n) = \sum_{m=0}^n a^{m-1}(a-m) \frac{n!}{(n-m)!m!} x^{\frac{n-m}{n}}. \quad (2.2)$$

A permutation $\sigma \in S_n$ is called a *cycle of length k* ($k = 1, 2, \dots, n$) if for k distinct points $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, n\}$, we have that $\sigma(i_j) = i_{j+1}$ ($j = 1, 2, \dots, k-1$), $\sigma(i_k) = i_1$ and $\sigma(i) = i$ for all $i \in \{1, 2, \dots, n\} \setminus \{i_1, i_2, \dots, i_k\}$. In the following, denote $l(\sigma)$ be the length of a cycle σ . It is well known (for example [6]) that each $\sigma \in S_n$ has a unique disjoint cycle decomposition $\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$, where each σ_i ($i = 1, 2, \dots, r$) is a cycle. In the following, for each $\sigma \in S_n$ with the unique disjoint cycle decomposition $\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$, we denote the maximal and the minimal length of σ_i ($i = 1, 2, \dots, r$) by $l_{\max}(\sigma)$ and $l_{\min}(\sigma)$ respectively, that is,

$$l_{\max}(\sigma) = \max\{l(\sigma_1), l(\sigma_2), \dots, l(\sigma_r)\},$$

and

$$l_{\min}(\sigma) = \min\{l(\sigma_1), l(\sigma_2), \dots, l(\sigma_r)\}.$$

Suppose that $k \in \{1, 2, \dots, n\}$. If $k|n$, it is not hard to see that τ_k^n (defined in (1.2)) can be decomposed into k disjoint cycles and each cycle has length $\frac{n}{k}$. For $k \nmid n$, if $(n, k) = r$, then $r|n$ and each $i \in \{1, 2, \dots, n\}$ can be written as $i = u + rv$, where $1 \leq u \leq r$ and $0 \leq v \leq \frac{n}{r} - 1$. Just as in [9], define $\sigma \in S_n$ by

$$\sigma(i) = \sigma(u + rv) \equiv u + kv \pmod{n}.$$

It is not hard to see that $\tau_r^n = \sigma^{-1} \tau_k^n \sigma$, that is, τ_r^n and τ_k^n are conjugate in S_n . Hence τ_r^n and τ_k^n have the same number of cycles of each type [6], that is, τ_k^n can be decomposed into r disjoint cycles and each cycle has length $\frac{n}{r}$. So for each $k \in \{1, 2, \dots, n\}$, we have that $l_{\min}(\tau_k^n) = l_{\max}(\tau_k^n) = \frac{n}{(n,k)}$. It is not hard to see that if $k \neq n$ and $\frac{n}{2}$ (when n ($n \geq 3$) is even), then $l_{\min}(\tau_k^n) = l_{\max}(\tau_k^n) \geq 3$. Hence we have the following lemma.

Lemma 2.3. *Suppose that $k \in \{1, 2, \dots, n\}$. Let τ_k^n be the permutation defined in (1.2). Then we have that $l_{\min}(\tau_k^n) = l_{\max}(\tau_k^n) = \frac{n}{(n,k)}$ and the following:*

- (i) if $k = n$, then $l_{\min}(\tau_k^n) = l_{\max}(\tau_k^n) = 1$;
- (ii) if $k = \frac{n}{2}$ when n is even, then $l_{\min}(\tau_k^n) = l_{\max}(\tau_k^n) = 2$;
- (iii) if $k \neq n$ and $\frac{n}{2}$ (when n ($n \geq 3$) is even), then $l_{\min}(\tau_k^n) = l_{\max}(\tau_k^n) \geq 3$.

Lemma 2.4. Let a, c_1, c_2, \dots, c_n be positive real numbers. For each $\sigma \in S_n$, if

$$a \geq \max\{n-1, n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}\}, \quad (2.3)$$

we have the following inequality

$$\frac{\alpha_1}{a\alpha_1 + c_1\alpha_{\sigma(1)}} + \frac{\alpha_2}{a\alpha_2 + c_2\alpha_{\sigma(2)}} + \frac{\alpha_3}{a\alpha_3 + c_3\alpha_{\sigma(3)}} + \cdots + \frac{\alpha_n}{a\alpha_n + c_n\alpha_{\sigma(n)}} \leq 1 \quad (2.4)$$

for any positive real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$. If σ is a cycle of length n , then the converse is also held.

Proof. Suppose that $a \geq \max\{n-1, n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}\}$. Let $x_i = c_i \frac{\alpha_{\sigma(i)}}{\alpha_i}$ for $i \in \{1, 2, \dots, n\}$, then $x_1 x_2 \cdots x_n = c_1 c_2 \cdots c_n$. For $F(x_1, \dots, x_n)$ in Lemma 2.1, we have

$$\begin{aligned} F(x_1, \dots, x_n) &= \sum_{m=0}^n \left[a^{m-1}(a-m) \sum_{1 \leq i_1 < \dots < i_{n-m} \leq n} x_{i_1} \cdots x_{i_{n-m}} \right] \\ &= \sum_{m=0}^{n-1} \left[a^{m-1}(a-m) \sum_{1 \leq i_1 < \dots < i_{n-m} \leq n} x_{i_1} \cdots x_{i_{n-m}} \right] + a^{n-1}(a-n) \\ &\geq \sum_{m=0}^{n-1} a^{m-1}(a-m) \frac{n!}{(n-m)!m!} (x_1 x_2 \cdots x_n)^{\frac{n-m}{n}} + a^{n-1}(a-n) \end{aligned} \quad (2.5)$$

$$\begin{aligned} &= \sum_{m=0}^n a^{m-1}(a-m) \frac{n!}{(n-m)!m!} (x_1 x_2 \cdots x_n)^{\frac{n-m}{n}} \\ &= ((c_1 c_2 \cdots c_n)^{\frac{1}{n}} + a)^{n-1} (a + (c_1 c_2 \cdots c_n)^{\frac{1}{n}} - n) \end{aligned} \quad (2.6)$$

$$\geq 0. \quad (2.7)$$

Since $a \geq n-1$, we have that $a-m \geq 0$ for $m = 0, 1, \dots, n-1$ and (2.5) is obtained by (2.1) in Lemma 2.2. From (2.2) of Lemma 2.2, we have (2.6). Since $a \geq n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}$, we have (2.7). Hence by Lemma 2.1 we get the desired inequality

$$\sum_{i=1}^n \frac{\alpha_i}{a\alpha_i + c_i\alpha_{\sigma(i)}} \leq 1.$$

Conversely, suppose that (2.4) holds for any positive real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and σ is a cycle of length n . It is not hard to see that $\{\sigma^1(1), \sigma^2(1), \dots, \sigma^n(1)\} = \{1, 2, \dots, n\}$.

First, we show that $a \geq n-1$. For any $\lambda > 0$, we choose

$$\alpha_i = \lambda^{-s} \quad \text{if } i = \sigma^s(1),$$

where $s \in \{1, 2, \dots, n\}$. Note that $1 = \sigma^n(1)$. So if $i = \sigma^s(1)$ with $s \in \{1, 2, \dots, n-1\}$, then for $i = 2, 3, \dots, n$ we have that $\sigma(i) = \sigma^{s+1}(1)$. Hence we have

$$\frac{\alpha_{\sigma(i)}}{\alpha_i} = \frac{\lambda^{-(s+1)}}{\lambda^{-s}} = \frac{1}{\lambda}, \quad i = 2, 3, \dots, n,$$

and

$$\frac{\alpha_{\sigma(1)}}{\alpha_1} = \frac{\lambda^{-1}}{\lambda^{-n}} = \lambda^{n-1}.$$

Now from (2.4) we have

$$\begin{aligned} \sum_{i=1}^n \frac{\alpha_i}{a\alpha_i + c_i\alpha_{\sigma(i)}} &= \frac{1}{a + c_1 \frac{\alpha_{\sigma(1)}}{\alpha_1}} + \frac{1}{a + c_2 \frac{\alpha_{\sigma(2)}}{\alpha_2}} + \cdots + \frac{1}{a + c_n \frac{\alpha_{\sigma(n)}}{\alpha_n}} \\ &= \frac{1}{a + c_1 \lambda^{n-1}} + \frac{1}{a + \frac{c_2}{\lambda}} + \cdots + \frac{1}{a + \frac{c_n}{\lambda}} \leq 1. \end{aligned} \quad (2.8)$$

Take $\lambda \rightarrow +\infty$, then we have that $\frac{n-1}{a} \leq 1$ by (2.8). So we obtain that $a \geq n-1$.

Next, we show that $a \geq n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}$. Let $d = (c_1 c_2 \cdots c_n)^{\frac{1}{n}}$. For each $i \in \{1, 2, \dots, n\}$, if $i = \sigma^k(1)$ for some $k \in \{1, 2, \dots, n\}$, we let

$$\alpha_i = \alpha_{\sigma^k(1)} = \frac{\alpha_1 d^k}{c_{\sigma^0(1)} c_{\sigma(1)} \cdots c_{\sigma^{k-1}(1)}}, \quad (2.9)$$

where $\sigma^0(1) = 1$.

For $i, k \in \{1, 2, \dots, n\}$, if $i = \sigma^k(1)$, then from (2.9) we have

$$\begin{aligned} c_i \frac{\alpha_{\sigma(i)}}{\alpha_i} &= c_{\sigma^k(1)} \frac{\alpha_{\sigma(\sigma^k(1))}}{\alpha_{\sigma^k(1)}} = c_{\sigma^k(1)} \frac{\alpha_{\sigma^{k+1}(1)}}{\alpha_{\sigma^k(1)}} \\ &= c_{\sigma^k(1)} \frac{\alpha_1 d^{k+1}}{c_{\sigma^0(1)} c_{\sigma(1)} \cdots c_{\sigma^k(1)}} \frac{c_{\sigma^0(1)} c_{\sigma(1)} \cdots c_{\sigma^{k-1}(1)}}{\alpha_1 d^k} \\ &= d. \end{aligned} \quad (2.10)$$

Hence from (2.10) and (2.4) we have

$$\sum_{i=1}^n \frac{\alpha_i}{a\alpha_i + c_i\alpha_{\sigma(i)}} = \sum_{i=1}^n \frac{1}{a + c_i \frac{\alpha_{\sigma(i)}}{\alpha_i}} = \frac{n}{a + d} \leq 1.$$

So we get $a \geq n - d = n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}$.

From discussions above, we have that $a \geq \max\{n-1, n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}\}$. \square

Lemma 2.5. ([23]) *Let A be a positive invertible operator on a Hilbert space, and ξ_0 the unit vector associated with a one dimensional projection P . Then $A \geq P$ if and only if $\langle A^{-1}\xi_0, \xi_0 \rangle \leq 1$.*

Theorem 2.6. *Let a, c_1, c_2, \dots, c_n be positive real numbers. For each $\sigma \in S_n$, if $a \geq \max\{n-1, n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}\}$, then $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n] : M_n \mapsto M_n$ is positive. Moreover, if σ is a cycle of length n , then the converse is also held.*

Proof. $\Theta^{(n,\sigma)}$ is positive if and only if $\Theta^{(n,\sigma)}(P) \geq 0$ for every one dimensional projection P , which means $\Delta^{(n,\sigma)}(P) \geq P$. Let $\xi_0 = (x_1, \dots, x_n)^t$ be the unit vector associated with P , that is, $P = \xi_0 \xi_0^*$. Without loss of generality, we can assume that $x_i \neq 0$ for $i = 1, \dots, n$. Then the matrix $\Delta^{(n,\sigma)}(P)$ has the form

$$\begin{pmatrix} a|x_1|^2 + c_1|x_{\sigma(1)}|^2 & 0 & \cdots & 0 \\ 0 & a|x_2|^2 + c_2|x_{\sigma(2)}|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a|x_n|^2 + c_n|x_{\sigma(n)}|^2 \end{pmatrix}.$$

Hence $A = \Delta^{(n,\sigma)}(P)$ is invertible and positive. From Lemma 2.5 we can see that $\Theta^{(n,\sigma)}$ is positive if and only if

$$\langle A^{-1}\xi_0, \xi_0 \rangle = \frac{|x_1|^2}{a|x_1|^2 + c_1|x_{\sigma(1)}|^2} + \frac{|x_2|^2}{a|x_2|^2 + c_2|x_{\sigma(2)}|^2} + \cdots + \frac{|x_n|^2}{a|x_n|^2 + c_n|x_{\sigma(n)}|^2}$$

$$\leq 1. \quad (2.11)$$

By Lemma 2.4 and (2.11), the proof is completed. \square

Remark 2.7. For $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$, suppose that $c \geq 0$, $a = n - c$ and $c_1 = c_2 = \dots = c_n = c$. The map $\Theta^{(n,\sigma)}[n - c; c, c, \dots, c]$ is discussed in Proposition 6.2 of [12]. For any $\sigma \in S_n$ which is not necessarily a cycle of length n , Hou, Li et al. showed that $\Theta^{(n,\sigma)}[n - c; c, c, \dots, c]$ is positive if and only if $c \leq \frac{n}{l_{\max}(\sigma)}$. In this case, we can see that there exists $\sigma \in S_n$ such that the positivity of $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ cannot imply “ $a \geq \max\{n - 1, n - (c_1 c_2 \dots c_n)^{\frac{1}{n}}\}$ ”. So for general $\sigma \in S_n$, it is interesting to find a necessary and sufficient condition for the positivity of $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$.

Suppose that $\phi : M_n \mapsto M_n$ is a linear map. For any positive integer k , let $M_k(M_n)$ denote the block matrix algebra of order k over M_n . Equivalently, $M_k(M_n)$ is often written as $M_k \otimes M_n$. Then we can define two linear maps ϕ_k and ϕ^k on $M_k \otimes M_n$ by

$$\phi_k((a_{ij})_{1 \leq i, j \leq k}) = (\phi(a_{ij}))_{1 \leq i, j \leq k}$$

and

$$\phi^k((a_{ij})_{1 \leq i, j \leq k}) = (\phi(a_{ij}^t))_{1 \leq i, j \leq k},$$

where $a_{ij} \in M_n$ for $i, j = 1, 2, \dots, k$. We say that ϕ is k -positive (or k -copositive) if ϕ_k (or ϕ^k) is positive. If ϕ_k (or ϕ^k) is positive for all $k = 1, 2, \dots$, then ϕ is said to be *completely positive* (or *completely copositive*).

The Choi matrix of a linear map $\psi : M_n \mapsto M_n$ is defined by

$$C_\psi = \sum_{i,j=1}^n E_{ij} \otimes \psi(E_{ij}) \in M_n \otimes M_n.$$

It is well known [1] that ψ is completely positive if and only if C_ψ is positive. It is not hard to see that ψ is completely copositive if and only if $\mathbf{T} \circ \psi$ is completely positive.

Theorem 2.8. *Let a, c_1, c_2, \dots, c_n be positive real numbers. For $\sigma \in S_n$, if $l_{\min}(\sigma) \geq 2$, then the following are equivalent:*

- (i) *the linear map $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is completely positive;*
- (ii) *the linear map $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is 2-positive;*
- (iii) *$a \geq n$.*

Proof. (i) \Rightarrow (ii) is clear by definition. For (ii) \Rightarrow (iii), we assume that $\Theta^{(n,\sigma)}$ is 2-positive. Let $\xi = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)^t \in \mathbb{C}^{2n}$ with $\|\xi\| = 1$. Let $x = (x_1, x_2, \dots, x_n)^t$, $y = (y_1, y_2, \dots, y_n)^t \in \mathbb{C}^n$. Then

$$P = \xi \xi^* = \begin{pmatrix} x x^* & x y^* \\ y x^* & y y^* \end{pmatrix} \in M_{2n}.$$

It is clear that P is a projection, and so we have $\Theta_2^{(n,\sigma)}(P) \geq 0$, that is,

$$\Delta_2^{(n,\sigma)}(P) \geq P. \quad (2.12)$$

For $\Delta_2^{(n,\sigma)}(P)$, we have

$$\Delta_2^{(n,\sigma)}(P) = \begin{pmatrix} \Delta^{(n,\sigma)}(x x^*) & \Delta^{(n,\sigma)}(x y^*) \\ \Delta^{(n,\sigma)}(y x^*) & \Delta^{(n,\sigma)}(y y^*) \end{pmatrix} = \sum_{i=1}^n A_i \otimes E_{ii}, \quad (2.13)$$

where

$$A_i = \begin{pmatrix} a|x_i|^2 + c_i|x_{\sigma(i)}|^2 & ax_i\bar{y}_i + c_i x_{\sigma(i)}\bar{y}_{\sigma(i)} \\ a\bar{x}_i y_i + c_i \bar{x}_{\sigma(i)} y_{\sigma(i)} & a|y_i|^2 + c_i|y_{\sigma(i)}|^2 \end{pmatrix} \in M_2$$

and $E_{ii} \in M_n$. Since $l_{\min}(\sigma) \geq 2$, we have that $\sigma(i) \neq i$ for each $i \in \{1, 2, \dots, n\}$, which means that σ has no fixed point. So for $i = 1, 2, \dots, n$, we can choose real numbers x_i, y_i such that each A_i is invertible. For example, we can choose $x_i = \alpha i$ and $y_i = \alpha$ where $\alpha = (\frac{n(n+1)(2n+1)}{6} + n)^{-\frac{1}{2}}$, that is,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n.$$

From the invertibility of each A_i , we see that $\Delta_2^{(n,\sigma)}(P)$ is invertible and

$$\Delta_2^{(n,\sigma)}(P)^{-1} = \sum_{i=1}^n A_i^{-1} \otimes E_{ii},$$

where

$$\begin{aligned} A_i^{-1} &= \begin{pmatrix} a|x_i|^2 + c_i|x_{\sigma(i)}|^2 & ax_i\bar{y}_i + c_i x_{\sigma(i)}\bar{y}_{\sigma(i)} \\ a\bar{x}_i y_i + c_i \bar{x}_{\sigma(i)} y_{\sigma(i)} & a|y_i|^2 + c_i|y_{\sigma(i)}|^2 \end{pmatrix}^{-1} \\ &= \frac{1}{ac_i\lambda_i} \begin{pmatrix} a|y_i|^2 + c_i|y_{\sigma(i)}|^2 & -(ax_i\bar{y}_i + c_i x_{\sigma(i)}\bar{y}_{\sigma(i)}) \\ -(a\bar{x}_i y_i + c_i \bar{x}_{\sigma(i)} y_{\sigma(i)}) & a|x_i|^2 + c_i|x_{\sigma(i)}|^2 \end{pmatrix} \end{aligned}$$

and $\lambda_i = |x_i y_{\sigma(i)} - x_{\sigma(i)} y_i|^2$.

Note that

$$\xi = \begin{pmatrix} x \\ y \end{pmatrix} = \sum_{i=1}^n z_i \otimes e_i,$$

where $z_i = (x_i \ y_i)^t \in \mathbb{C}^2$. So we obtain

$$\begin{aligned} \langle A_i^{-1} z_i, z_i \rangle &= \frac{1}{ac_i\lambda_i} \left\langle \begin{pmatrix} a|y_i|^2 + c_i|y_{\sigma(i)}|^2 & -(ax_i\bar{y}_i + c_i x_{\sigma(i)}\bar{y}_{\sigma(i)}) \\ -(a\bar{x}_i y_i + c_i \bar{x}_{\sigma(i)} y_{\sigma(i)}) & a|x_i|^2 + c_i|x_{\sigma(i)}|^2 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right\rangle \\ &= \frac{c_i\lambda_i}{ac_i\lambda_i} = \frac{1}{a}. \end{aligned} \tag{2.14}$$

By Lemma 2.5, (2.12) and (2.14), we have

$$\begin{aligned} \langle \Delta_2^{(n,\sigma)}(P)^{-1} \xi, \xi \rangle &= \left\langle \left(\sum_{i=1}^n A_i^{-1} \otimes E_{ii} \right) \left(\sum_{j=1}^n z_j \otimes e_j \right), \sum_{j=1}^n z_j \otimes e_j \right\rangle \\ &= \sum_{i=1}^n \langle A_i^{-1} z_i, z_i \rangle = \sum_{i=1}^n \frac{1}{a} \\ &= \frac{n}{a} \leq 1. \end{aligned}$$

So $a \geq n$, and (iii) holds.

Assume that (iii) holds. Since $l_{\min}(\sigma) \geq 2$, it is not hard to see that the eigenfunction of $C_{\Theta^{(n,\sigma)}}$ is

$$g(\lambda) = \det(\lambda I_{n^2} - C_{\Theta^{(n,\sigma)}}) = \lambda^{n^2-2n}(\lambda - a)^{n-1}(\lambda - a + n) \prod_{i=1}^n (\lambda - c_i).$$

If $a \geq n$, the eigenvalues of $C_{\Theta^{(n,\sigma)}}$ are nonnegative. So $\Theta^{(n,\sigma)}$ is completely positive and (i) holds. \square

In Proposition 6.3 of [12], Hou, Li et al. gave similar results as Theorem 2.8 above. For the D -type linear map Λ_D discussed there, all row sums and column sums of the nonnegative matrix D associated to Λ_D are equal to n . In Theorem 2.8 above, we have not required that.

For $\sigma = \tau_{n-1}^n$ ($n \geq 2$), from Lemma 2.3 we see that τ_{n-1}^n is a cycle of length n , and so $l_{\min}(\sigma) = n$. Hence we obtain Theorem 2.5 of [7] from Theorem 2.8 above. If $\sigma = id_n$, then we have that $l_{\min}(id_n) = 1$. In this case, we have the following result.

Proposition 2.9. *For any positive numbers a, c_1, \dots, c_n , the following conditions are equivalent:*

(i) *the matrix*

$$A = \begin{pmatrix} a + c_1 - 1 & -1 & \cdots & -1 \\ -1 & a + c_2 - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & a + c_n - 1 \end{pmatrix}$$

is positive;

(ii) $\Theta^{(n, id_n)}[a; c_1, c_2, \dots, c_n]$ *is positive;*

(iii) $\Theta^{(n, id_n)}[a; c_1, c_2, \dots, c_n]$ *is completely positive.*

Proof. Suppose that $\sigma = id_n$ and $X \in M_n$. By the definition of $\Theta^{(n, \sigma)}$, we can see that

$$\Theta^{(n, id_n)}(X) = A * X,$$

where $A * X$ denotes the Schur product of A and X . Hence, using Theorem 3.7 in [18], we get the equivalence of (i), (ii) and (iii). \square

For general $\sigma \in S_n$ with $l_{\min}(\sigma) = 1$, the situation becomes more complicated. In [20], Qi and Hou defined a linear map $\Delta_{(t_1, t_2, \dots, t_n)}$ in some more general environment. The following result improves Proposition 2.7 in [20].

Corollary 2.10. *Let H and K be Hilbert spaces and let $\{f_i\}_{i=1}^n$ and $\{f'_i\}_{i=1}^n$ be any orthonormal sets of H and K , respectively. Let $F_{ji} = f'_j f_i^* \in B(H, K)$ be a rank one operator such that for any $x \in H$ we have $F_{ji}(x) = \langle x, f_i \rangle f'_j$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on H . Let $\Delta_{(t_1, t_2, \dots, t_n)} : B(H) \mapsto B(K)$ be defined by*

$$\Delta_{(t_1, t_2, \dots, t_n)}(X) = \sum_{i=1}^n t_i F_{ii} X F_{ii}^* - \left(\sum_{i=1}^n F_{ii} \right) X \left(\sum_{i=1}^n F_{ii} \right)^* \quad (2.15)$$

for all $X \in B(H)$. Then the following conditions are equivalent:

(i) *the matrix*

$$A = \begin{pmatrix} t_1 - 1 & -1 & \cdots & -1 \\ -1 & t_2 - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & t_n - 1 \end{pmatrix}$$

is positive;

(ii) $\Delta_{(t_1, t_2, \dots, t_n)}$ *is positive;*

(iii) $\Delta_{(t_1, t_2, \dots, t_n)}$ *is completely positive.*

Proof. Since $\Delta_{(t_1, t_2, \dots, t_n)}$ is a finite rank elementary operator [20], it is not hard to see that if we let $t_i = a + c_i$ for $i = 1, 2, \dots, n$ we can identify it with $\Theta^{(n, id_n)}[a; c_1, c_2, \dots, c_n]$. By Proposition 2.9, we obtain the equivalence of (i), (ii) and (iii). \square

3. ATOMICITY AND DECOMPOSABILITY

In this section we discuss when $\Theta^{(n,\sigma)}$ is atomic and decomposable. Let $\phi : M_n \mapsto M_n$ be a linear map. In [17], Osaka defined a real linear map $\tilde{\phi} : M_n(\mathbb{R}) \mapsto M_n(\mathbb{R})$ by

$$\tilde{\phi}(x) = \frac{1}{2} (\phi(x) + \overline{\phi(x)}), \quad x = (x_{ij}) \in M_n(\mathbb{R}),$$

where $\overline{(y_{ij})} = (\overline{y_{ij}})$ for $y = (y_{ij}) \in M_n$. It is not hard to see that if ϕ is k -positive or k -copositive, then so is $\tilde{\phi}$ for $k = 1, 2, \dots$. The following lemma indicates that when $k = 2$ and $l_{\min}(\sigma) \geq 2$ the converse is also true for $\Theta^{(n,\sigma)}$.

Lemma 3.1. *Let a, c_1, c_2, \dots, c_n be positive real numbers. For $\sigma \in S_n$ with $l_{\min}(\sigma) \geq 2$, if $\tilde{\Theta}^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is 2-positive, then $a \geq n$, and so $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is 2-positive.*

Proof. It is clear that $\Theta^{(n,\sigma)}(x) = \tilde{\Theta}^{(n,\sigma)}(x)$ for $x \in M_n(\mathbb{R})$. Suppose that $\tilde{\Theta}^{(n,\sigma)}$ is 2-positive. In the proof of Theorem 2.8, for $i = 1, 2, \dots, n$ we can choose real numbers x_i, y_i such that each A_i is invertible. Thus if we apply the proof of Theorem 2.8 to $\tilde{\Theta}^{(n,\sigma)}$, we can also get that $a \geq n$. So $\Theta^{(n,\sigma)}$ is 2-positive by Theorem 2.8. \square

Lemma 3.2. *Suppose that $\sigma \in S_n$ ($n \geq 3$) and $l_{\min}(\sigma) \geq 3$. Let $\phi : M_n \mapsto M_n$ be a positive linear map. Suppose that $\{\phi(E_{ij})\}_{i,j=1}^n$ satisfy the following conditions:*

- (i) $\phi(E_{ii})e_j = e_j^*\phi(E_{ii}) = 0$ for each $1 \leq i \leq n$ and $j \in \{1, 2, \dots, n\} \setminus \{i, \sigma^{-1}(i)\}$;
- (ii) $\phi(E_{ij}) = -E_{ij}$ for $1 \leq i \neq j \leq n$.

If $\phi = \varphi + \psi$, where φ is a 2-positive linear map and ψ is a 2-copositive linear map, then $\tilde{\phi} : M_n(\mathbb{R}) \mapsto M_n(\mathbb{R})$ is a 2-positive linear map.

Proof. First, we show that $\psi(E_{ij})$ is a diagonal matrix for $i \neq j$. Since φ is 2-positive and ψ is 2-copositive, we have

$$\begin{pmatrix} \varphi(E_{ii}) & \varphi(E_{ij}) \\ \varphi(E_{ji}) & \varphi(E_{jj}) \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} \psi(E_{ii}) & \psi(E_{ji}) \\ \psi(E_{ij}) & \psi(E_{jj}) \end{pmatrix} \geq 0. \quad (3.1)$$

Let $j \in \{1, 2, \dots, n\} \setminus \{i, \sigma^{-1}(i)\}$ and $1 \leq i \leq n$. By condition (i), we have

$$\begin{aligned} \langle \phi(E_{ii})e_j, e_j \rangle &= \langle (\varphi(E_{ii}) + \psi(E_{ii}))e_j, e_j \rangle \\ &= e_j^*\phi(E_{ii})e_j = 0. \end{aligned}$$

Using the positivity of φ and ψ , we obtain that $e_j^*\varphi(E_{ii})e_j = 0$ and $e_j^*\psi(E_{ii})e_j = 0$. Hence φ and ψ also satisfy condition (i).

Note that if $\begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix} \in M_2$ is positive and $x = 0$ or $z = 0$, then we must have $y = 0$; any principal submatrix of a positive matrix must be a positive matrix. So from condition (i), we can see that the nonzero elements in the $n \times n$ matrices $\varphi(E_{ii})$ and $\psi(E_{ii})$ can only appear in these positions: (i, i) , $(i, \sigma^{-1}(i))$, $(\sigma^{-1}(i), i)$ and $(\sigma^{-1}(i), \sigma^{-1}(i))$. From (3.1), for $i \neq j$ we can see that the nonzero elements of the $n \times n$ matrix $\varphi(E_{ij})$ can only appear in the positions: (i, j) , $(i, \sigma^{-1}(j))$, $(\sigma^{-1}(i), j)$ and $(\sigma^{-1}(i), \sigma^{-1}(j))$; the nonzero elements of the $n \times n$ matrix $\psi(E_{ij})$ can only appear in the positions: (j, i) , $(j, \sigma^{-1}(i))$, $(\sigma^{-1}(j), i)$ and $(\sigma^{-1}(j), \sigma^{-1}(i))$. Hence for $1 \leq i, j \leq n$ we have

$$\begin{aligned} \varphi(E_{ij}) &= y_{ij}E_{ij} + y_{i,\sigma^{-1}(j)}E_{i,\sigma^{-1}(j)} + y_{\sigma^{-1}(i),j}E_{\sigma^{-1}(i),j} \\ &\quad + y_{\sigma^{-1}(i),\sigma^{-1}(j)}E_{\sigma^{-1}(i),\sigma^{-1}(j)} \end{aligned}$$

and

$$\psi(E_{ij}) = z_{ji}E_{ji} + z_{\sigma^{-1}(j),i}E_{\sigma^{-1}(j),i} + z_{j,\sigma^{-1}(i)}E_{j,\sigma^{-1}(i)}$$

$$+ z_{\sigma^{-1}(j), \sigma^{-1}(i)} E_{\sigma^{-1}(j), \sigma^{-1}(i)}, \quad (3.2)$$

where all y 's and z 's above are complex numbers.

For $i \neq j$, by condition (ii) we have

$$\begin{aligned} \phi(E_{ij}) &= \varphi(E_{ij}) + \psi(E_{ij}) \\ &= y_{ij} E_{ij} + y_{i, \sigma^{-1}(j)} E_{i, \sigma^{-1}(j)} + y_{\sigma^{-1}(i), j} E_{\sigma^{-1}(i), j} + y_{\sigma^{-1}(i), \sigma^{-1}(j)} E_{\sigma^{-1}(i), \sigma^{-1}(j)} + \\ &\quad z_{ji} E_{ji} + z_{\sigma^{-1}(j), i} E_{\sigma^{-1}(j), i} + z_{j, \sigma^{-1}(i)} E_{j, \sigma^{-1}(i)} + z_{\sigma^{-1}(j), \sigma^{-1}(i)} E_{\sigma^{-1}(j), \sigma^{-1}(i)} \\ &= -E_{ij}. \end{aligned} \quad (3.3)$$

If $\psi(E_{ij}) = 0$, then clearly $\psi(E_{ij})$ is diagonal. Suppose that $\psi(E_{ij}) \neq 0$. Since $\{E_{ij}\}_{1 \leq i, j \leq n}$ are linear independent, by comparing indices in (3.3) it can only happen that

- (1) $\sigma^{-1}(j) = i$ and $\sigma^{-1}(i) \neq j$;
- (2) $\sigma^{-1}(i) = j$ and $\sigma^{-1}(j) \neq i$;
- (3) $\sigma^{-1}(j) = i$ and $\sigma^{-1}(i) = j$;
- (4) $\sigma^{-1}(j) = j$ or $\sigma^{-1}(i) = i$.

Suppose that condition (1) holds. From (3.3) it is not hard to see that $z_{\sigma^{-1}(j), i} = -y_{i, \sigma^{-1}(j)} \neq 0$ and $z_{ji} = z_{j, \sigma^{-1}(i)} = z_{\sigma^{-1}(j), \sigma^{-1}(i)} = 0$. So from (3.2) we can see that $\psi(E_{ij})$ is diagonal. Similarly, $\psi(E_{ij})$ is also diagonal if condition (2) holds.

Since $l_{\min}(\sigma) \geq 3$, condition (3) and condition (4) cannot happen. If condition (3) holds, then $\sigma(j) = i$ and $\sigma(i) = j$. Thus there exists a cycle of length 2 in the disjoint cycle decomposition of σ . So we have $l_{\min}(\sigma) \leq 2$ which is contradict to our assumption. Similarly, we can see that condition (4) cannot happen. Thus we can see that $\psi(E_{ij})$ are diagonal matrices for all $1 \leq i \neq j \leq n$. Hence $\psi(E_{ij})^t = \psi(E_{ij})$ for all $1 \leq i \neq j \leq n$.

Next, we show that $\tilde{\psi}$ is 2-positive. Since ψ is positive, $\psi(x^*) = \psi(x)^*$ for any $x \in M_n$. From discussions above, we have

$$\psi(E_{ij}) = \psi(E_{ji}^*) = \psi(E_{ji})^* = \overline{\psi(E_{ji})^t} = \overline{\psi(E_{ji})} \quad \text{for all } 1 \leq i \neq j \leq n; \quad (3.4)$$

$$\psi(E_{ii}) = \psi(E_{ii})^* = \overline{\psi(E_{ii})^t} \quad \text{for } 1 \leq i \leq n. \quad (3.5)$$

For each $(x_{ij}) \in M_n(\mathbb{R})$, from (3.4) and (3.5) we have

$$\begin{aligned} \tilde{\psi}((x_{ij})) &= \frac{1}{2} \left(\psi((x_{ij})) + \overline{\psi((x_{ij}))} \right) = \frac{1}{2} \left(\sum_{i,j=1}^n x_{ij} \psi(E_{ij}) + \sum_{i,j=1}^n x_{ij} \overline{\psi(E_{ij})} \right) \\ &= \frac{1}{2} \left(\sum_{i=1}^n x_{ii} (\psi(E_{ii}) + \psi(E_{ii})^t) + \sum_{1 \leq i \neq j \leq n} x_{ij} (\psi(E_{ij}) + \psi(E_{ji})) \right) \end{aligned}$$

and

$$\begin{aligned} \tilde{\psi}((x_{ij})^t) &= \frac{1}{2} \left(\psi((x_{ji})) + \overline{\psi((x_{ji}))} \right) = \frac{1}{2} \left(\sum_{i,j=1}^n x_{ji} \psi(E_{ij}) + \sum_{i,j=1}^n x_{ji} \overline{\psi(E_{ij})} \right) \\ &= \frac{1}{2} \left(\sum_{i=1}^n x_{ii} (\psi(E_{ii}) + \psi(E_{ii})^t) + \sum_{1 \leq i \neq j \leq n} x_{ji} (\psi(E_{ij}) + \psi(E_{ji})) \right). \end{aligned}$$

So we have

$$\tilde{\psi}(X) = \tilde{\psi}(X^t), \quad (3.6)$$

for any $X \in M_n(\mathbb{R})$.

Now for each $\begin{pmatrix} X & Y \\ Y^t & Z \end{pmatrix} \geq 0$ in $M_2(M_n(\mathbb{R}))$, we have

$$\begin{aligned} \tilde{\psi}_2 \begin{pmatrix} X & Y \\ Y^t & Z \end{pmatrix} &= \begin{pmatrix} \tilde{\psi}(X) & \tilde{\psi}(Y) \\ \tilde{\psi}(Y^t) & \tilde{\psi}(Z) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}(X) & \tilde{\psi}(Y^t) \\ \tilde{\psi}(Y) & \tilde{\psi}(Z) \end{pmatrix} \\ &= \tilde{\psi}^2 \begin{pmatrix} X & Y \\ Y^t & Z \end{pmatrix} \geq 0, \end{aligned}$$

where the second equality is followed from (3.6), and the last inequality is followed from the 2-copositivity of $\tilde{\psi}$. So $\tilde{\psi}$ is 2-positive.

Since $\tilde{\phi} = \tilde{\varphi} + \tilde{\psi}$ and both $\tilde{\varphi}$ and $\tilde{\psi}$ are 2-positive, we have that $\tilde{\phi}$ is 2-positive. \square

Suppose that $\phi : M_n \mapsto M_n$ is a positive linear map. ϕ is said to be *atomic* if ϕ can not be decomposed into a sum of a 2-positive map and a 2-copositive map. If ϕ can be decomposed into sums of completely positive maps and completely copositive maps, then ϕ is said to be *decomposable*, otherwise, ϕ is said to be *indecomposable*. Let 1_k denote the identity map on M_k and \mathbf{T} denote the transpose map on M_n , the *partial transpose* X^Γ of a matrix X in $M_k \otimes M_n$ is defined by

$$X^\Gamma = (1_k \otimes \mathbf{T})(X).$$

It is not hard to see that ϕ is decomposable if and only if C_ϕ can be decomposed as sums of positive matrices and matrices whose partial transpose are positive.

Theorem 3.3. *Let a, c_1, c_2, \dots, c_n ($n \geq 3$) be positive real numbers. Suppose that $\sigma \in S_n$ and $l_{\min}(\sigma) \geq 3$. If $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is positive but not completely positive, then $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is atomic. Particularly, if $n > a \geq \max\{n-1, n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}\}$, then $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is atomic.*

Proof. Since $l_{\min}(\sigma) \geq 3$ and $\Theta^{(n,\sigma)}$ is positive but not completely positive, we have that $a < n$ by Theorem 2.8.

Assume that $\Theta^{(n,\sigma)} = \varphi + \psi$, where φ is 2-positive and ψ is 2-copositive. For $1 \leq i, j \leq n$, we have

$$\Theta^{(n,\sigma)}(E_{ij}) = \begin{cases} (a-1)E_{ii} + c_{\sigma^{-1}(i)}E_{\sigma^{-1}(i),\sigma^{-1}(i)} & \text{if } i = j \\ -E_{ij} & \text{if } i \neq j \end{cases}.$$

Hence $\{\Theta^{(n,\sigma)}(E_{ij})\}_{i,j=1}^n$ satisfy conditions in Lemma 3.2, and so $\tilde{\Theta}^{(n,\sigma)}$ is 2-positive. By Lemma 3.1, we have that $a \geq n$ which is a contradiction. Hence $\Theta^{(n,\sigma)}$ is atomic.

Particularly, if $n > a \geq \max\{n-1, n - (c_1 c_2 \cdots c_n)^{\frac{1}{n}}\}$, from Theorem 2.6 and Theorem 2.8 we can see that $\Theta^{(n,\sigma)}$ is positive but not completely positive. Thus $\Theta^{(n,\sigma)}$ is atomic. \square

Remark 3.4. For $\sigma \in S_n$, if $\sigma^2 = id_n$, then the lengths of cycles in the disjoint cycle decomposition of σ are not greater than 2, that is, $l_{\max}(\sigma) \leq 2$ and $l_{\min}(\sigma) \leq 2$. In Proposition 7.2 of [12], Hou, Li et al. showed that if $\sigma^2 = id_n$, then $\Theta^{(n,\sigma)}[n-1; 1, 1, \dots, 1]$ is decomposable. In Proposition 3.7 below, for $\sigma^2 = id_n$ we also obtain a class of decomposable maps of the form $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$. Hence for $l_{\min}(\sigma) \leq 2$, there exist positive linear maps of the form $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ which are decomposable and hence not atomic.

Since $(n, n-1) = 1$, from Lemma 2.3 we have that $l_{\min}(\tau_{n-1}^n) = n$. In Theorem 3.3, if we let $\sigma = \tau_{n-1}^n$, then we obtain Theorem 3.2 in [7]. For $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ ($c \geq 0$), the condition when it is positive and completely positive was discussed in [12]. In the following corollary, we give conditions when it is atomic.

Corollary 3.5. *Suppose that $\sigma \in S_n$ and $0 \leq c \leq \frac{n}{l_{\max}(\sigma)}$. If $l_{\min}(\sigma) \geq 3$ and $0 < c \leq \frac{n}{l_{\max}(\sigma)}$, then $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is atomic. If $c = 0$, then $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is completely positive.*

Proof. If $0 < c \leq \frac{n}{l_{\max}(\sigma)}$ and $l_{\min}(\sigma) \geq 3$, then by Proposition 6.2 of [12] we have that $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is positive. Since $l_{\min}(\sigma) \geq 3$ and $n-c < n$, by Theorem 2.8 we can see that $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is positive but not completely positive. Thus by theorem 3.3, $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is atomic.

If $c = 0$, then we can see that $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ takes the form of (2.15). By Corollary 2.10, it is not hard to see that $\Theta^{(n,\sigma)}[n; 0, 0, \dots, 0]$ is completely positive. \square

It is clear that $\Theta^{(n,\tau_k^n)}[n-1; 1, 1, \dots, 1]$ ($n \geq 3, k \in \{1, 2, \dots, n-1\}$) is the map $\Phi^{(k)}$, defined in [20] if we restrict $\Phi^{(k)}$ to M_n . In [20], Qi and Hou showed that if $k \neq \frac{n}{2}$, then $\Phi^{(k)}$ is indecomposable. Here we give the following result.

Corollary 3.6. *For each $k \in \{1, 2, \dots, n-1\}$ ($n \geq 3$), if $k \neq \frac{n}{2}$ when n is even, then $\Theta^{(n,\tau_k^n)}[n-1; 1, 1, \dots, 1]$ is atomic.*

Proof. Suppose that $k \in \{1, 2, \dots, n-1\}$ ($n \geq 3$) and $k \neq \frac{n}{2}$ when n is even. By Lemma 2.3, we have that $l_{\min}(\tau_k^n) \geq 3$. In Theorem 3.3, if we let $a = n-1$ and $c_1 = c_1 = \dots = c_n = 1$, then we can see that $\Theta^{(n,\tau_k^n)}[n-1; 1, 1, \dots, 1]$ is atomic. \square

The following proposition extends Proposition 7.2 in [12].

Proposition 3.7. *Suppose that $\sigma \in S_n$ and $\sigma^2 = id_n$. Let $F = \{i : \sigma(i) = i, i = 1, 2, \dots, n\}$ and $F^c = \{1, 2, \dots, n\} \setminus F$. Let a, c_1, c_2, \dots, c_n be positive real numbers. If $a \geq n-1$, $c_i \geq 1$ when $i \in F$ and $c_i c_{\sigma(i)} \geq 1$ when $i \in F^c$, then $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is decomposable.*

Proof. Let

$$P = \sum_{i \in F} (a + c_i - 1) E_{ii} \otimes E_{ii} + (a - 1) \sum_{i \in F^c} E_{ii} \otimes E_{ii} - \sum_{\substack{1 \leq i \neq j \leq n \\ \sigma(i) \neq j}} E_{ij} \otimes E_{ij}.$$

Note that P is unitarily equivalent to $B \oplus 0$, where $B = (b_{ij}) \in M_n$ is a Hermitian matrix satisfying: $b_{ii} = a - 1$ or $a + c_i - 1$; $b_{ij} = 0$ or -1 (when $i \neq j$). Since $a \geq n-1$ and $c_i \geq 1$ when $i \in F$, we can see that B is a diagonally dominant Hermitian matrix. From the well-known strictly diagonal dominance theorem [10], it is not hard to see that B is positive. Therefore, P is positive.

Since $\sigma^2 = id_n$, the lengths of cycles in the disjoint cycle decomposition of σ are not greater than 2. By definition we know that if $i \in F^c$, then $\sigma(i) \in F^c$, $i \neq \sigma(i)$ and $(i, \sigma(i))$ is a cycle of length 2. So the number k of elements in F^c is even. Then F^c consists of $\frac{k}{2}$ pairs of elements and each pair is of the form $(i, \sigma(i))$. For $i, \sigma(i) \in F^c$, without loss of generality we assume that $i < \sigma(i)$, and denote

$$Q_i = c_{\sigma(i)} E_{ii} \otimes E_{\sigma(i), \sigma(i)} + c_i E_{\sigma(i), \sigma(i)} \otimes E_{ii} - E_{i, \sigma(i)} \otimes E_{i, \sigma(i)} - E_{\sigma(i), i} \otimes E_{\sigma(i), i}.$$

For $i \in F^c$, since $c_i c_{\sigma(i)} \geq 1$, it is not hard to see that the partial transpose Q_i^Γ of Q_i in $M_n \otimes M_n$ is positive.

For $i, j \in \{1, 2, \dots, n\}$, since $\sigma^2 = id_n$, by definition we have

$$\Theta^{(n,\sigma)}(E_{ii}) = (a - 1) E_{ii} + c_{\sigma^{-1}(i)} E_{\sigma^{-1}(i), \sigma^{-1}(i)} = (a - 1) E_{ii} + c_{\sigma(i)} E_{\sigma(i), \sigma(i)}$$

and

$$\Theta^{(n,\sigma)}(E_{ij}) = -E_{ij}, \quad i \neq j.$$

It is not hard to see that the Choi matrix of $\Theta^{(n,\sigma)}$ is

$$\begin{aligned} C_{\Theta^{(n,\sigma)}} &= \sum_{i,j=1}^n E_{ij} \otimes \Theta^{(n,\sigma)}(E_{ij}) \\ &= P + \sum_{\substack{i \in P^c \\ i < \sigma(i)}} Q_i. \end{aligned}$$

So the Choi matrix of $\Theta^{(n,\sigma)}$ is the sums of positive matrices and matrices whose partial transposes are positive. From the correspondence between positive linear maps and Choi matrices discussed before, we see that $\Theta^{(n,\sigma)}$ is decomposable. \square

4. SEPARABILITY OF STRUCTURAL PHYSICAL APPROXIMATIONS AND OPTIMALITY OF ENTANGLEMENT WITNESSES

In this final section, as applications we give conditions to ensure the separability of the structural physical approximation of $\Theta^{(n,\sigma)}$ and the optimality of the entanglement witness associated with $\mathbf{T} \circ \Theta^{(n,\sigma)}$.

Let ϕ be a nonzero positive linear map of M_n into itself. Since $Tr(C_\phi) = Tr(\phi(I_n))$, we see that $Tr(C_\phi) > 0$. Let $W = \frac{1}{Tr(C_\phi)} C_\phi$. Since W is a Hermitian matrix, there are $W^+, W^- \geq 0$ such that $W^+ W^- = 0$ and $W = W^+ - W^-$, and similarly for C_ϕ if we put $C_\phi^+ = Tr(C_\phi) W^+$ and $C_\phi^- = Tr(C_\phi) W^-$. For $0 \leq \lambda \leq 1$, let

$$\tilde{W}(\lambda) = \frac{1-\lambda}{n^2} I_n \otimes I_n + \lambda W.$$

By equation (14) of [5],

$$\lambda^* = \frac{1}{1 + n^2 \|W^-\|}$$

is the maximal λ such that $\tilde{W}(\lambda) \geq 0$.

In [22], Stormer gave a formula of structural physical approximation for unital linear maps of M_n into itself. Generally, let ϕ be any nonzero positive linear map of M_n into itself. The structural physical approximation of ϕ (denoted by $SPA(\phi)$) is defined as

$$\begin{aligned} SPA(\phi) &= \tilde{W}(\lambda^*) = \frac{1}{n^2} \left(1 - \frac{1}{1 + n^2 \|W^-\|} \right) I_n \otimes I_n + \frac{1}{1 + n^2 \|W^-\|} W \\ &= \frac{\|W^-\|}{1 + n^2 \|W^-\|} I_n \otimes I_n + \frac{1}{1 + n^2 \|W^-\|} W \\ &= \frac{Tr(C_\phi)^{-1} \|C_\phi^-\|}{1 + n^2 Tr(C_\phi)^{-1} \|C_\phi^-\|} I_n \otimes I_n + \frac{Tr(C_\phi)^{-1}}{1 + n^2 Tr(C_\phi)^{-1} \|C_\phi^-\|} C_\phi \\ &= \frac{1}{Tr(C_\phi) + n^2 \|C_\phi^-\|} (\|C_\phi^-\| I_n \otimes I_n + C_\phi). \end{aligned} \tag{4.1}$$

Recall that a positive matrix $A \in M_m \otimes M_n$ is said to be *separable* if $A = \sum_{i=1}^k B_i \otimes C_i$ for some $k \in \mathbb{N}$, and positive matrices $B_i \in M_m$ and $C_i \in M_n$ for $i = 1, 2, \dots, k$. In the following proposition, if we let $c_1 = c_2 = \dots = c_n = 1$ and $\sigma = \tau_k^n$ ($k = 1, 2, \dots, n-1$ and $k \neq \frac{n}{2}$ when n is even), we obtain Proposition 4.2 in [21].

Proposition 4.1. *Suppose that $\sigma \in S_n$ and $l_{\min}(\sigma) \geq 2$. For positive real numbers a, c_1, c_2, \dots, c_n , if $a = n-1$ and $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is positive, then the structural physical approximation of $\Theta^{(n,\sigma)}[a; c_1, c_2, \dots, c_n]$ is separable.*

Proof. For $\sigma \in S_n$, if $l_{\min}(\sigma) \geq 2$ and $a = n - 1$, it is not hard to see that $C_{\Theta(n,\sigma)}$ is unitarily equivalent to $G \oplus H$, where

$$G = \begin{pmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & n-2 \end{pmatrix} \in M_n$$

and $H \in M_{n^2-n}$ is a diagonal matrix whose diagonal consists of c_i ($i = 1, 2, \dots, n$) and 0. Since G has only one negative eigenvalue: -1 , so is $C_{\Theta(n,\sigma)}$. Thus we have $\|C_{\Theta(n,\sigma)}^-\| = 1$. Since $\text{Tr}(C_{\Theta(n,\sigma)}) = n(n-2) + \sum_{i=1}^n c_i$, by (4.1) we have

$$\begin{aligned} SPA(C_{\Theta(n,\sigma)}) &= \frac{1}{\text{Tr}(C_{\Theta(n,\sigma)}) + n^2 \|C_{\Theta(n,\sigma)}^-\|} (\|C_{\Theta(n,\sigma)}^-\| I_n \otimes I_n + C_{\Theta(n,\sigma)}) \\ &= \frac{1}{\text{Tr}(C_{\Theta(n,\sigma)}) + n^2 \|C_{\Theta(n,\sigma)}^-\|} (I_n \otimes I_n + C_{\Theta(n,\sigma)}) \\ &= \frac{1}{n(n-2) + \sum_{i=1}^n c_i + n^2} \left(\sum_{i,j=1}^n E_{ii} \otimes E_{jj} + (n-2) \sum_{i=1}^n E_{ii} \otimes E_{ii} \right. \\ &\quad \left. + \sum_{i=1}^n c_{\sigma^{-1}(i)} E_{ii} \otimes E_{\sigma^{-1}(i), \sigma^{-1}(i)} - \sum_{1 \leq i \neq j \leq n} E_{ij} \otimes E_{ij} \right). \end{aligned} \quad (4.2)$$

It is not hard to see that

$$\begin{aligned} &\sum_{i,j=1}^n E_{ii} \otimes E_{jj} + (n-2) \sum_{i=1}^n E_{ii} \otimes E_{ii} - \sum_{1 \leq i \neq j \leq n} E_{ij} \otimes E_{ij} \\ &= \sum_{1 \leq i < j \leq n} (E_{ii} \otimes E_{ii} + E_{jj} \otimes E_{jj} + E_{ii} \otimes E_{jj} + E_{jj} \otimes E_{ii} - E_{ij} \otimes E_{ij} - E_{ji} \otimes E_{ji}). \end{aligned}$$

Let $\sigma_{ij} = E_{ii} \otimes E_{ii} + E_{jj} \otimes E_{jj} + E_{ii} \otimes E_{jj} + E_{jj} \otimes E_{ii} - E_{ij} \otimes E_{ij} - E_{ji} \otimes E_{ji}$. To illustrate the separability of $\sigma_{ij} \in M_n \otimes M_n$, in this paragraph we let $\{e_i^{(2)} : i = 1, 2, \dots, n\}$ and $\{e_i^{(n)} : i = 1, 2, \dots, n\}$ denote the canonical orthonormal basis of \mathbb{C}^2 and \mathbb{C}^n , respectively. Let $\{E_{ij}^{(2)} : i, j = 1, 2\}$ denote the canonical matrix units of M_2 . Let

$$\begin{aligned} R &= E_{11}^{(2)} \otimes E_{11}^{(2)} + E_{11}^{(2)} \otimes E_{22}^{(2)} + E_{22}^{(2)} \otimes E_{11}^{(2)} \\ &\quad + E_{22}^{(2)} \otimes E_{22}^{(2)} - E_{12}^{(2)} \otimes E_{12}^{(2)} - E_{21}^{(2)} \otimes E_{21}^{(2)} \\ &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \in M_2 \otimes M_2. \end{aligned}$$

Let R^Γ be the partial transpose of R in $M_2 \otimes M_2$. It is not hard to see that R and R^Γ are positive. From Theorem 2 of [11] we can see that a positive matrix in $M_2 \otimes M_2$ is separable if and only if its partial transpose is positive, hence R is separable. Let

$$D_{ij} = (e_i^{(n)} e_1^{(2)*} + e_j^{(n)} e_2^{(2)*}) \in M_{n \times 2},$$

where $e_1^{(2)*}$ and $e_2^{(2)*}$ denote the conjugate transpose of $e_1^{(2)}$ and $e_2^{(2)}$, respectively. Note that $\sigma_{ij} = (D_{ij} \otimes D_{ij}) R (D_{ij}^* \otimes D_{ij}^*)$. Since R is separable, we have that σ_{ij} is separable.

From (4.2) we have

$$SPA(C_{\Theta^{(n,\sigma)}}) = \frac{1}{n(n-2) + \sum_{i=1}^n c_i + n^2} \left(\sum_{1 \leq i < j \leq n} \sigma_{ij} + \sum_{i=1}^n c_{\sigma^{-1}(i)} E_{ii} \otimes E_{\sigma^{-1}(i), \sigma^{-1}(i)} \right).$$

Hence $SPA(C_{\Theta^{(n,\sigma)}})$ is separable. \square

Let $\phi : M_n \mapsto M_n$ be a positive linear map. If ϕ is not completely positive, then

$$W_\phi = \frac{1}{n} C_\phi$$

is called the *entanglement witness* associated to ϕ . An entanglement witness is said to be *optimal* if it detects a maximal set of entanglement [15]. It was shown in [15] that if W_ϕ has the *spanning property*, that is, $\mathcal{P}_{W_\phi} = \{\zeta : \langle W_\phi \zeta, \zeta \rangle = 0, \text{ where } \zeta = \xi \otimes \eta \in \mathbb{C}^n \otimes \mathbb{C}^n\}$ spans the whole space $\mathbb{C}^n \otimes \mathbb{C}^n$, then W_ϕ is an optimal entanglement witness.

For $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ which was discussed in Proposition 6.2 of [12], let $c = 1$ and $\sigma = \tau_k^n$ ($k = 1, 2, \dots, n$). It was shown in [8] that if $k \neq n$ and $\frac{n}{2}$ (when n ($n \geq 3$) is even), then the entanglement witness associated to $\mathbf{T} \circ \Theta^{(n,\tau_k^n)}[n-1; 1, 1, \dots, 1]$ is optimal. Using the method in [8], in the following we give conditions when the entanglement witness associated to $\mathbf{T} \circ \Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is optimal.

Theorem 4.2. Suppose that $\sigma \in S_n$ ($n \geq 3$) and $l_{\min}(\sigma) \geq 3$. If $0 < c \leq \frac{n}{l_{\max}(\sigma)}$, then the entanglement witness associated to $\mathbf{T} \circ \Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is optimal.

Proof. Suppose that $\sigma \in S_n$ and $l_{\min}(\sigma) \geq 3$. Since $0 < c \leq \frac{n}{l_{\max}(\sigma)}$, by Corollary 3.5 we know that $\Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is positive. Since \mathbf{T} is positive, $\mathbf{T} \circ \Theta^{(n,\sigma)}[n-c; c, c, \dots, c]$ is also positive.

For $i, j \in \{1, 2, \dots, n\}$, since $\Theta^{(n,\sigma)}(E_{ii}) = (n-c-1)E_{ii} + cE_{\sigma^{-1}(i), \sigma^{-1}(i)}$ and $\Theta^{(n,\sigma)}(E_{ij}) = -E_{ij}$ ($i \neq j$), we have

$$\begin{aligned} W_{\mathbf{T} \circ \Theta^{(n,\sigma)}} &= \frac{1}{n} C_{\mathbf{T} \circ \Theta^{(n,\sigma)}} = \frac{1}{n} \left(\sum_{i,j=1}^n E_{ij} \otimes \mathbf{T} \circ \Theta^{(n,\sigma)}(E_{ij}) \right) \\ &= \frac{1}{n} \left(\sum_{i,j=1}^n E_{ij} \otimes W_{ji}^{(n,\sigma)} \right), \end{aligned}$$

where

$$W_{ij}^{(n,\sigma)} = \begin{cases} (n-c-1)E_{ii} + cE_{\sigma^{-1}(i), \sigma^{-1}(i)} & \text{if } i = j \\ -E_{ij} & \text{if } i \neq j \end{cases}.$$

For any n -tuple $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ of real numbers θ_j , let

$$\mathcal{S} = \{\xi_\theta \otimes \xi_\theta : \xi_\theta = \sum_{j=1}^n e^{i\theta_j} e_j\}. \quad (4.3)$$

For each $i \in \{1, 2, \dots, n\}$, let

$$\mathcal{V}_i^{(n,\sigma)} = \{e_i \otimes e_j : j \neq i \text{ and } j \neq \sigma^{-1}(i), \text{ where } 1 \leq j \leq n\}. \quad (4.4)$$

For $\eta_\theta = \xi_\theta \otimes \xi_\theta \in \mathcal{S}$ where $\xi_\theta = \sum_{j=1}^n e^{i\theta_j} e_j$, we have

$$\langle \eta_\theta W_{\mathbf{T} \circ \Theta^{(n,\sigma)}}, \eta_\theta \rangle = \frac{1}{n} \left(\sum_{i=1}^n (\xi_\theta \otimes \xi_\theta)^* (E_{ii} \otimes ((n-c-1)E_{ii} + cE_{\sigma^{-1}(i), \sigma^{-1}(i)})) (\xi_\theta \otimes \xi_\theta) \right)$$

$$\begin{aligned}
& - \sum_{1 \leq i \neq j \leq n} (\xi_\theta \otimes \xi_\theta)^* E_{ij} \otimes E_{ji} (\xi_\theta \otimes \xi_\theta) \Big) \\
& = \frac{1}{n} \left((n-c-1) \sum_{j=1}^n |e^{i\theta_j}|^2 |e^{i\theta_j}|^2 + c \sum_{j=1}^n |e^{i\theta_j}|^2 |e^{i\theta_{\sigma^{-1}(j)}}|^2 - \sum_{1 \leq j \neq l \leq n} |e^{i\theta_j}|^2 |e^{i\theta_l}|^2 \right) \\
& = \frac{1}{n} \left((n-c-1)n + nc - (n^2 - n) \right) \\
& = 0.
\end{aligned}$$

Suppose that $l, m \in \{1, 2, \dots, n\}$ and $f_{lm} = e_l \otimes e_m \in \mathcal{V}_l^{(n, \sigma)}$. Since $m \neq l$ and $m \neq \sigma^{-1}(l)$, we have

$$\begin{aligned}
\langle f_{lm} W_{\mathbf{T} \circ \Theta^{(n, \sigma)}}, f_{lm} \rangle & = \frac{1}{n} \left(\sum_{i=1}^n (e_l \otimes e_m)^* \left(E_{ii} \otimes ((n-c-1)E_{ii} + cE_{\sigma^{-1}(i), \sigma^{-1}(i)}) \right) (e_l \otimes e_m) \right. \\
& \quad \left. - \sum_{1 \leq i \neq j \leq n} (e_l \otimes e_m)^* E_{ij} \otimes E_{ji} (e_l \otimes e_m) \right) \\
& = 0.
\end{aligned}$$

Now we will show that if $l_{\min}(\sigma) \geq 3$, the vectors defined in (4.3) and (4.4) span the whole space $\mathbb{C}^n \otimes \mathbb{C}^n$. For each vector $\sum_{i,j=1}^n x_i y_j e_i \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n$, it can be identified with a matrix $\sum_{i,j=1}^n x_i y_j E_{ij} \in M_n$. So we identify $\mathbb{C}^n \otimes \mathbb{C}^n$ with M_n . In [8], Ha showed that the vectors $\xi_\theta \otimes \xi_\theta$ in (4.3) span all symmetric matrices E_{ii} and $E_{ij} + E_{ji}$ ($1 \leq i \neq j \leq n$) in M_n under the identification between $\mathbb{C}^n \otimes \mathbb{C}^n$ and M_n .

For $1 \leq i, j \leq n$, if $i \neq j$, we have either $e_i \otimes e_j \in \mathcal{V}_i^{(n, \sigma)}$ or $e_j \otimes e_i \in \mathcal{V}_j^{(n, \sigma)}$. If not, by the definition of $\{\mathcal{V}_i^{(n, \sigma)}\}_{i=1}^n$, we have that $i = \sigma^{-1}(j)$ and $j = \sigma^{-1}(i)$, that is, $\sigma(i) = j$ and $\sigma(j) = i$. So there is a cycle of length 2 in the disjoint cycle decomposition of σ which contradicts the assumption $l_{\min}(\sigma) \geq 3$. Thus under the identification between $\mathbb{C}^n \otimes \mathbb{C}^n$ and M_n , either E_{ij} or E_{ji} ($1 \leq i \neq j \leq n$) lies in the linear span of vectors in (4.4).

From discussion above, we can see that the vectors in (4.3) and (4.4) span the whole space $\mathbb{C}^n \otimes \mathbb{C}^n$. Hence $W_{\mathbf{T} \circ \Theta^{(n, \sigma)}}$ has the spanning property, and so it is optimal. \square

Suppose that $k \in \{1, 2, \dots, n-1\}$, $n \geq 3$ and $k \neq \frac{n}{2}$ when n is even. From Lemma 2.3, we have $l_{\min}(\tau_k^n) \geq 3$. In Theorem 4.2, if we let $\sigma = \tau_k^n$ and $c = 1$, then we obtain Theorem 1 in [8]. In Theorem 4.2, if we let $c = 0$, then $\Theta^{(n, \sigma)}[n-c; c, c, \dots, c] = \Theta^{(n, \sigma)}[n; 0, 0, \dots, 0] = \Delta_{(n, n, \dots, n)}$, where $\Delta_{(n, n, \dots, n)}$ is defined in (2.15). For $\Delta_{(n, n, \dots, n)}$, we have the following proposition.

Proposition 4.3. *Suppose that $n \geq 2$. Then $\mathbf{T} \circ \Delta_{(n, n, \dots, n)}$ is decomposable and the entanglement witness associated to $\mathbf{T} \circ \Delta_{(n, n, \dots, n)}$ is optimal.*

Proof. From Corollary 3.5, we have that $\Delta_{(n, n, \dots, n)}$ is completely positive. So $\mathbf{T} \circ \Delta_{(n, n, \dots, n)}$ is decomposable.

Let $W_{\mathbf{T} \circ \Delta_n}$ be the entanglement witness associated to $\mathbf{T} \circ \Delta_{(n, n, \dots, n)}$. For each $i \in \{1, 2, \dots, n\}$ and $n \geq 2$, let

$$\mathcal{V}_i^{(n)} = \{e_i \otimes e_j : j \neq i, \text{ where } 1 \leq j \leq n\}.$$

For $n \geq 2$, just as (4.3), let

$$\mathcal{S} = \{\xi_\theta \otimes \xi_\theta : \xi_\theta = \sum_{j=1}^n e^{i\theta_j} e_j\},$$

where θ_j ($j = 1, 2, \dots, n$) are arbitrary real numbers. Just as the proof of Theorem 4.2, it is not hard to check that the vectors in $\{\cup_{i=1}^n \mathcal{V}_i^{(n)}\} \cup \mathcal{S}$ span the whole space $\mathbb{C}^n \otimes \mathbb{C}^n$ and $\langle W_{T \circ \Delta_n} \xi, \xi \rangle = 0$ for each $\xi \in \{\cup_{i=1}^n \mathcal{V}_i^{(n)}\} \cup \mathcal{S}$. Thus $W_{T \circ \Delta_n}$ has the spanning property, and so it is optimal. \square

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